Math 245C Lecture 20 Notes

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1 Fourier Inversion

1.1 Fourier transform of exponentials

For $a > 0$, recall that

$$f_a^n(x) = e^{-\pi a|x|^2} = \prod_{j=1}^n f_a^1(x_j).$$

Additionally,

$$\int_{\mathbb{R}} \frac{e^{-|x_j-u|^2/2\theta}}{\sqrt{2\pi\theta}}\, dx_j = 1 \implies \int_{\mathbb{R}^n} \frac{e^{-|x-u|^2/2\theta}}{\sqrt{2\pi\theta}}\, dx = 1.$$

Lemma 1.1. *We have*

$$\widehat{f_a^n} = \frac{1}{\sqrt{a}^n} f_{1/a}^n.$$

Proof. Note that

$$\begin{aligned}\widehat{f_a^n}(\xi) = \int_{\mathbb{R}^n} e^{-2\pi i \xi\cdot x} f_a^n(x)\, dx &= \prod_{j=1}^n \int_{\mathbb{R}} e^{-2\pi i \xi_j x_j} f_a^1(x_j)\, dx_j \\ &= \prod_{j=1}^n \widehat{f_a^1}(\xi_j).\end{aligned}$$

So it suffices to show the lemma for $n = 1$. Assume $n = 1$.

We want to show that

$$e^{(\pi/a)\xi^2}\widehat{f_a^1}(\xi) = 1.$$

We claim that for $f = f_a^1$,

$$\frac{d}{d\xi}\left(\widehat{f}(\xi)e^{(\pi/a)\xi^2}\right) = 0.$$

We have

$$\frac{d}{d\xi}\widehat{f} = -\widehat{2\pi i x f} = \frac{i}{a}\widehat{2\pi x a e^{-\pi a|x|^2}} = \frac{i}{a}\widehat{\frac{d}{dx}(e^{-\pi a|x|^2})} = -\frac{i}{a}\widehat{\frac{df}{dx}} = -\frac{i}{a}(-2\pi i \xi)\widehat{f} = -\frac{2\pi}{a}\xi \widehat{f}(\xi).$$

Hence,

$$\begin{aligned}\frac{d}{d\xi}(\widehat{f}(\xi)e^{(\pi/a)\xi^2}) &= \frac{d}{d\xi}\widehat{f}(\xi)e^{(\pi/a)\xi^2} + \widehat{f}(\xi)\frac{2\pi}{a}\xi e^{-\pi a\xi^2}\\ &= \xi\frac{2\pi}{a}\widehat{f}(\xi)\left[-e^{(\pi/a)\xi^2} + e^{(\pi/a)\xi^2}\right]\\ &= 0.\end{aligned}$$

Consequently,

$$e^{(\pi/a)\xi^2}\widehat{f_a^1}(\xi) = \widehat{f}(0) = \int_{\mathbb{R}} f(x)\,dx = \int_{\mathbb{R}} e^{-\pi x^2 a}\,dx = \frac{1}{\sqrt{a}}.$$ □

1.2 Self-adjoint property of the Fourier transform

Lemma 1.2. *Let $f, g \in L^1$. Then*

$$\int_{\mathbb{R}^n} \widehat{f} g\,d\xi = \int_{\mathbb{R}^n} f\widehat{g}\,dx.$$

Proof. We have

$$\begin{aligned}\int_{\mathbb{R}^n} f(x)\widehat{g}(x)\,dx &= \int_{\mathbb{R}^n} f(x)\int_{\mathbb{R}^n} e^{-2\pi i\xi\cdot x}g(\xi)\,d\xi\,dx\\ &= \int_{\mathbb{R}^n\times\mathbb{R}^n} f(x)g(\xi)e^{-2\pi i\xi\cdot x}\,dx\,d\xi\\ &= \int_{\mathbb{R}^n\times\mathbb{R}^n} f(\xi)g(x)e^{-2\pi i\xi\cdot x}\,dx\,d\xi\\ &= \int_{\mathbb{R}^n} \widehat{f}(\xi)g(\xi)\,d\xi.\end{aligned}$$ □

1.3 The Fourier inversion formula

Definition 1.1. Let $F \in L^1$. We define

$$F^\vee = \widehat{F}(-\xi) = \int_{\mathbb{R}^n} e^{2\pi i\xi\cdot x}F(x)\,dx.$$

Theorem 1.1. *Suppose $F, \widehat{F} \in L^1$. There exists $G \in C_0$ such that $F = G$ a.e. and $(F^\vee)^\wedge = (\widehat{F})^\vee = G$.*

Proof. For each $x \in \mathbb{R}^n$ and $t > 0$, define

$$\phi_t^x(\xi) = e^{2\pi i\xi\cdot x - \pi|\xi|^2 t} = E_x(\xi)f_t^n(\xi).$$

Note that

$$\widehat{\phi_t^x}(y) = \langle E_x f_t^n, E_y \rangle = \langle f_t^n, E_{y-x} \rangle = \widehat{f}_t(y-x) = \frac{1}{\sqrt{t}^n} f_{1/t}^n(y-x) = \frac{1}{\sqrt{t}^n} f_{1/t}^n(x-y).$$

by the lemma. But setting $t = 2\pi\theta$ gives

$$\int_{\mathbb{R}^n} f_{1/t}^n(z) \frac{1}{\sqrt{t}^n} \, dz = 1.$$

In conclusion, $\phi_t^x, \widehat{\phi_t^x} \in L^1$. By the previous lemma,

$$\int_{\mathbb{R}^n} \widehat{\phi_t^x} F(\xi) \, d\xi = \int_{\mathbb{R}^n} \phi_t^x(y) \widehat{F}(y) \, d\xi.$$

Using the expression of $\widehat{\phi_t^x}$, we obtain

$$\int_{\mathbb{R}^n} \frac{1}{\sqrt{t}^n} f_{1/t}^n(x-\xi) F(\xi) \, d\xi = \int_{\mathbb{R}^n} \widehat{F}(y) e^{2\pi i y \cdot x - \pi |y|^2 t} \, dy.$$

Hence

$$\lim_{t \to 0} \rho_t * F = \lim_{t \to 0} \int_{\mathbb{R}^n} \widehat{F}(y) e^{2\pi i y \cdot x - \pi |y|^2 t} \, dy, \qquad \text{where } \rho_t(z) = \frac{1}{\sqrt{t}^n} f_{1/t}^n(z).$$

By the dominated convergence theorem,

$$\lim_{t \to 0} \rho_t * F = \int_{\mathbb{R}^n} \widehat{F}(y) e^{2\pi i y \cdot x} \, dx = (\widehat{F})^\vee(-x),$$

and $F = \lim_{t \to 0} \rho_t * F$ a.e., as

$$\rho_t(z) = \frac{1}{\sqrt{t}^n} e^{-\pi |z/\sqrt{t}|^2} = \frac{1}{\sqrt{t}^n} \rho_1(z/\sqrt{t}).$$

So we have proven that

$$F(x) = (\widehat{F})^\wedge(-x) = (\widehat{F})^\vee(x) \qquad a.e.$$

We have shown that $(\widehat{F})^\wedge = (F \circ O)$, where $O(z) = -z$. Now $F^\vee = \widehat{F} \circ 0$, so $(F^\vee)^\wedge = (\widehat{F} \circ O)^\wedge$. □